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## S–4223

## M.A./M.Sc. (Fourth Semester) Examination, 2024-25

## MATHEMATICS

## Paper – XXII

## [Measure and Integration]

## (MATH-C-016)

**Time : Two Hours] [Maximum Marks : 60**

**Note :** Attempt any *four* questions. All questions carry equal marks.

**1.** (a) Prove that measure of Cantor's ternary set is zero.

(b) If $<E_n>$ is countable collection of sets, then prove that :

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n).$$

**2.** Prove that a countable union of measurable sets is a measurable set.

**3.** (a) Let $f$ and $g$ be measurable functions on a measurable set E, then prove that $f/g$ is also measurable, where $g \neq 0$ on E.

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(b) Prove that a function is measurable if and only if both its positive and negative are measurable functions.

4. If a sequence $<f_n>$ converges in measure to $f$ on E, then prove that there exists a subsequence $<f_{n_k}>$ of $<f_n>$ which converges to $f$ a.e. on E.

5. (a) If $f$ is a bounded function defined on $[a, b]$ and $f$ is R-integrable on $[a, b]$, then prove that f is also L-integrable on $[a, b]$.

(b) If $f$ is a bounded measurable function on a measurable subset E on $[a, b]$ then prove that $|f|$ is L-integrable on E and $\left|\int_E f\right| \leq \int_E |f|$.

6. (a) Let $<f_n>$ be an increasing sequence of non-negative measurable functions, and let $\lim_{n\to\infty} f_n = f$ on E, then prove that $\lim_{n\to\infty} \int_E f_n = \int_E f$.

(b) Using Lebesgue dominated convergence theorem, evaluate :

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx, \text{ where } f_n(x) = \frac{n^{3/2} \cdot x}{1 + n^2 \cdot x^2}, \; n = 1, 2, 3, \ldots, 0 \leq x \leq 1.$$

7. Let $f$ be a function of bounded variation on $[a, b]$, then prove that $f$ is continous at a point in $[a, b]$ if and only if its variation function $v_f$ is so.

8. Let $f$ be an integrable function on $[a, b]$ and suppose

$$F(x) = \int_a^x f(t).dt + F(a).$$

Then prove that $F'(x) = f(x)$ a.e. in $[a, b]$.