Roll No. ....

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S–1138

M. A./M. Sc. (First Semester) EXAMINATION, 2023-24

MATHEMATICS

Paper Fourth

(Complex Analysis)

(MATH–004)

Time : Two Hours] [*Maximum Marks : 60*

Note : Attempt any *four* questions. Each question carries 15 marks.

1. (a) Prove that the function :

$$\sin\left[z+z^{-1}\right]=\sum_{n=0}^{\infty} a_n z^n+\sum_{n=1}^{\infty} b_n z^{-n}$$

where $a_n = b_n = \frac{1}{2\pi}\int_0^{2\pi}\sin(2\cos\theta)\cos n\theta\, d\theta$.

(b) State the Laurent's series and find Laurent's series of $f(z)=\frac{1}{z(z-1)}$, valid in the region $1<|z-2|<2$.

P. T. O.

2. (a) Define the radius of convergence of a power series and find radius of convergene of the following series :

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n+2)(n!)^2}(z-i)^{2n}$$

(b) State and prove the Cauchy residue theorem.

3. Using Cauchy-Residue theorem, evaluate the following integrals :

(i) $\int_0^{2\pi} \frac{1}{1+\sin^2\theta} d\theta$

(ii) $\int_0^{\infty} \frac{x \sin x}{x^2+4} dx$

4. Evaluate :

$$\int_0^{\infty} \frac{\sin \pi x}{x\left(1-x^2\right)} dx$$

using Cauchy-Residue theorem.

5. (a) To prove that for each point z of a domain, where $f(z)$ is analytic and $f'(z) \neq 0$, the mapping $w = f(z)$ is conformal.

(b) Find the image of :

$$R : x=0, y=0, x=2, y=1$$

rectangular region under the mapping :

$$w = \sqrt{2}.e^{i\frac{\pi}{4}}.z + (1+i)$$

6. (a) Show that $w = iz + i$ maps half plane $x > 0$ onto half plane $v > 1$.

(b) Show that the transform $w = \dfrac{2z+3}{z-4}$, transforms the circle $x^2 + y^2 - 4x = 0$ into the straight line $4u + 3 = 0$.

7. (a) State the Rouche's theorem and using it locate the roots of the polynomial :

$$g(z) = z^{10} - 6z^7 + 3z^3 - 1$$

(b) Define analytic continuation and show that :

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2n+1} \text{ and } g(z) = \sum_{n=0}^{\infty} \frac{(z-1)^n}{(2-i)^{n+1}}$$

are analytic continuation of each other.

8. State and prove Mittag Leffler's expansion theorem.